M.Sc. - II (Mathematics) (NEP Pattern) Semester-IV

04NEPMATH02 - Major : Functional Analysis

P. Pages : 2 **GUG/S/25/16359**

Time : Three Hours *3953* Max. Marks : 80

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT - I

1. a) Show that the space R of all real numbers is a normed linear space with norm x of a real number be defined by $\|x\| = |x|$. Also, show that R is a Banach space **8**

b) If M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M, then prove that there exist a function f_0 in N^* such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$. **8**

OR

c) Let M be a linear subspace of a normed linear space N, and let F be a functional defined on M. Then prove that F can be extended to a functional F_0 defined on the whole space N such that $\|F_0\| = \|F\|$. **8**

d) Prove that if N is a normed linear space, then the closed unit sphere S* in N* is a compact Hausdorff space in the weak* topology. **8**

UNIT - II

2. a) Prove that: A non-empty subset X of a normed linear space N is bounded $\Leftrightarrow f(X)$ is a bounded set of numbers for each f in N*. **8**

b) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm. **8**

OR

c) If $B \& B'$ are Banach spaces and if T is a linear transformation of B into B', then prove that T is continuous iff its graph is closed. **8**

d) Let M be a closed linear subspace of a Hilbert space H, let x be a vector not in M, and let d be the distance from x to M. Then prove that there exists a unique vector y_0 in M such that $\|x - y_0\| = d$. **8**

UNIT - III

3. a) If T is an operator on H for which $(Tx, x) = 0$ for all x, then prove that T=0. **8**

b) Prove that the real Banach space of all self-adjoint operator on H is a partially ordered set whose linear structure and order structure are related by the following properties- **8**

i) If $A_1 \le A_2$, then prove that $A_1 + A \le A_2 + A$ for every A;

ii) If $A_1 \le A_2$ and $\alpha \ge 0$, then prove that $\alpha A_1 \le \alpha A_2$.

OR

c) If N_1 and N_2 are normal operators on H with the property that either commutes with the adjoint of the other, then prove that their $N_1 + N_2$ and $N_1 N_2$ are normal. **8**

d) If T is an operator on H, then prove that T is normal $\Leftrightarrow$ it's real and imaginary parts commute, **8**

UNIT - IV

4. a) Let B be basis for H, and T an operator whose matrix relative to B is $\left[\alpha_{ij}\right]$. Then prove that T is non-singular $\Leftrightarrow \left[\alpha_{ij}\right]$ is non-singular, and in this case $\left[\alpha_{ij}\right]^{-1} = \left[T^{-1}\right]$. **8**

b) Show that the dimension of algebra $\mathscr{B}(H)$ is n^2. **8**

OR

c) If T is an arbitrary operator on H, then prove that the eigen values of T constitute a non-empty finite subset of the complex plane. Furthermore, show that the number of points in this set does not exceed the dimension n of the space H. **8**

d) Prove that- **8**

i) If T is normal, then each M_i reduces T.

ii) If T is normal, then the M_i's span H.

5. a) Prove that norm is a continuous function. **4**

b) Define- **4**

i) Orthogonal set   ii) Orthogonal complement of a set

c) Prove that an operator T on H is normal $\Leftrightarrow \|T^*x\| = \|Tx\|$ for every x. **4**

d) Let T be an arbitrary operator on H; and N a normal operator. Show that If T commutes with N, then T also commutes with N*. **4**

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